Space of Quantum Theory Representations of Natural Numbers, Integers, and Rational Numbers

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Abstract

This paper extends earlier work on quantum theory representations of natural numbers N, integers I, and rational numbers Ra to describe a space of these representations and transformations on the space. The space is parameterized by 4-tuple points in a parameter set. Each point, (k, m, h, g), labels a specific representation of $X = N,\ I,\ Ra$ as a Fock space $\mathcal{F}^X_{k,m,h}$ of states of finite length strings of qukits q and a basis set $\mathcal{B}_{k,m,h,g}$ of states of q strings, denoted together as $\mathcal{FB}^X_{k,m,h,g}$. The pair (m, h) locates the q strings in a square integer lattice $I \times I$. k denotes the qukit base, and g fixes the gauge or basis for the states of each q. Basic arithmetic relations and operations are described for the states in $\mathcal{FB}^X_{k,m,h,g}$. Maps $(k, m, h, g) \to (k', m', h', g')$ induce transformations $\mathcal{FB}^X_{k,m,h,g} \to \mathcal{FB}^X_{k',m',h',g'}$ on the representation space. There are two shifts, a base change operator $W_{k',k}$, and a gauge transformation function U_k where $U_k(j, h)$ is an element of the unitary group $U(k)$. Invariance of the axioms and theorems for N, I, and Ra under any transformation is discussed. It is seen that the properties of $W_{k',k}$ depend on the prime factors of k' and k. This suggests that one consider prime number $q's$, $q_2,\ q_3,\ q_5$, etc. as elementary and the base k $q's$ as composites of the prime number $q's$.

1 Introduction

It is quite evident that numbers play a fundamental role both in experimental and theoretical physics and in much of mathematics. There are several different types of numbers all of which are important. Natural numbers are the most basic as they are used for counting. Integers extend natural numbers to include negative whole numbers. Rational numbers are essential to both theory and

experiment as they are used in most theoretical computations. They also appear as the output of experiments. Real and complex numbers are the basis of all physical theories and of many mathematical theories. Also all theoretical predictions in physics can be cast in the form of real number solutions to equations.

By themselves these reasons provide support for investigation of the properties of numbers and their representations and their relation to physics. However, there is also an underlying motivation for this and related work. This is the need to understand why mathematics is so effective and relevant to physics.

This problem, which was expressed by Wigner in 1960 [1] and discussed by others [2, 3], is particularly acute if one accepts the widely held Platonic view that mathematical objects have some type of ideal, abstract existence with properties that are true or false in some ideal sense [8, 9]. If physical existence of systems refers to systems that both exist in and determine the properties of space time, then there is no reason why mathematics and physics should be related at all. Yet it is obvious that they are very closely related.

There are several different approaches to understanding this relationship [4]-[7]. The approach underlying this paper is to work towards a coherent theory of physics and mathematics together [10]. Such a theory, by treating both physics and mathematics together in one theory, would be expected to describe both physical and mathematical systems and how they are related, possibly as complementary aspects of a more basic type of system. It may also help to answer some of the basic outstanding questions in physics.

The method used here is to study properties of numbers with an emphasis on some concepts that are important to physics. Numbers are chosen because they are so basic to physics, as measurement outputs, as theoretical predictions and as computer outputs. The representations used here, as states of *single*, finite length strings of qukits, are based on the observations that all physical representations of numbers are by single strings of digits and that quantum theory is the basic theory of all physical systems. In addition, the use of quantum rather than classical representations brings both the treatment of physical systems and numbers into the same general theory. The use of the same basic theory to describe both physical systems and numbers as mathematical systems should help in bringing together descriptions of physical and mathematical systems

The use of quantum theory to study representations of numbers and other mathematical systems is not new [11] -[19]. Of particular note is work on quantum set theory represented as an orthomodular lattice valued set theory [15]-[19]. In this work natural numbers, integers, and rational numbers have representations that are either similar to the usual ones in mathematical analysis [15]-[18] or are based on a categorical approach [17, 19]. However the work in these references differs from the approach taken here in that numbers are represented here as states of finite qukit strings.

This work extends other work on quantum representations of numbers for a binary base [20], including those studied in quantum computing [22], to descriptions for all bases $k \geq 2$. The study is limited to quantum representations of natural numbers, N, integers, I, and rational numbers Ra. Extension to

quantum representations of real and complex numbers will be treated in future work[1]

The description of each quantum representation as a space of states of finite qukit strings, and spaces of these representations, is based on a parameterization of the representations and an association of a specific representation to each point in the parameter set. The points in the set are represented by quadruples $(k, (m, h), g)$. The integer pair (m, h) locates the string on a 2 dimensional integer lattice, $I \times I$, k is a natural number ≥ 2 that is the number base, and g is a gauge fixing function on $N \geq 2 \times I \times I$. For each integer pair (j, h) and each $k \geq 2$, $g(k, j, h)$ is a basis choice for a k dimensional Hilbert space at (j, h).

Transformations $(k, (m, h), g) \to (k', (m', h'), g')$ in the parameter set induce transformations in the representation space. These consist of unitary translations that move the qukit strings on the lattice, transformations that change states of strings of base k qukits to states of strings of base k' qukits, and unitary gauge transformations for each k. These are maps from the lattice points to elements of U_k.

An interesting result is that the axioms and theorems for each of the three types of numbers are invariant under these transformations. They represent symmetries of the systems. This is the case even though the specific expressions of the axioms and theorems in terms of basic arithmetic relations and operations are different for different representations. This is like the situation in physics where the laws of physics are invariant under Lorentz transformations even though their specific expression in different reference frames may be different.

Another interesting result is that qukits q_k where k is a prime number function as elementary qukits. These are the "elementary particles" as far as quantum representations of numbers are concerned. Qukits where k is not prime can be considered as composites of the prime number q_k. This follows from the properties of the k changing operator, especially for rational number representations.

The plan of the paper is as follows: In the next section quantum representations of the natural numbers, integers, and rational numbers by states of strings of base k qukits are discussed. Here k is the number of internal states of each qukit.

Section 3 and its subsections describe the parameter set and the induced space of quantum theory representations of numbers. Properties of transformations in the representation space associated with transformations of the parameter set are discussed in some detail. Of special note are the dependence of the base change operations on the number type (N, I, Ra) and on the base values. Commutation relations for the transformations are also discussed as are the special properties of unary representations corresponding to $k = 1$.

Section 4 discusses symmetries and invariances associated with the transformations. The idea is that, since axioms and theorems hold for each representation, they are invariant under any transformation. As such they are symmetries

[1]Earlier work on quantum representations of real and complex numbers have been limited to states based on strings of qubits [21].

of the space.

Composite and elementary qukits are discussed in the next section. It is noted that base k qukits where k is a prime number, play the role of elementary qukits. This is a consequence of the properties of sets of Ra represented by states of single strings of kits or qukits. The prime number qukits can be combined into composites to give $q_k's$ for any base k. The paper concludes with a discussion of several points and a summary of what was done here.

It is to be emphasized that all numbers are represented here as states of *single*, finite length strings of qukits. This is based on the observation that almost all physical representations of numbers as experimental outputs and as used in computers are of this form. Representations of numbers by pairs of states of two finite length qukit strings are not used here.

2 Quantum Representations of Natural Numbers, Integers, and Rational Numbers

Here representations of N, I, and Ra are given as states of finite strings of base k qukits on an integer lattice $I \times I$. The description that follows is an extension of earlier work [21, 23] on qubit strings to qukit strings for any $k \geq 2$.

Since states for strings of different numbers of qukits are needed, it is useful to describe the states using strings of annihilation creation (AC) operators that create or annihilate qukits in different states at different integer pair locations. Two types of qukits are used, $(a_k)_{\alpha,(j,h)}, (a_k^\dagger)_{\alpha,(j,h)}$ and $c_{\gamma,(m,h)}, c^\dagger_{\gamma,(m,h)}$. Here $\alpha = 0, 1, \cdots, k-1$ $\gamma = +, -$, and j, m, h are integers. There is no k subscript on the c operators as they are the same qubit operators for all k values. For each value of k, the AC operators can satisfy either commutation relations or anticommutation relations as the base k qukits can be either bosons or fermions. For all k the type a and c systems are assumed to be distinguishable.

Points on the integer lattice on which the qukits q_k are located consist of pairs (j, h) of integers. Each value of h gives the location of a string and the values of j give the locations of the q_k in a string. Figure 1 shows schematic examples of two q_k strings on the lattice, one at h and the other at $h + 1$.

It should be emphasized that it is not necessary here to assume that $I \times I$ denotes discrete points in a metric space. Nothing is assumed here about any metric spatial distance between points or even if such exists. The only feature of $I \times I$ used here is that it represents a pair of integer type orderings: Each I is a collection of points with no least and no greatest point and each point has a nearest neighbor on each side.[2]

A compact representation of numbers is used that is suitable for representing the three types of numbers. Here the integer location of the sign is also the location of the "k-al" point. For example -63.71, 459, -0.0753 would be

[2]Even this is more than is needed. It is sufficient for one I to be ordered to give an ordering to the q_k in a string. The other I could be replaced by a denumerable set of labels that serve to distinguish the different strings.

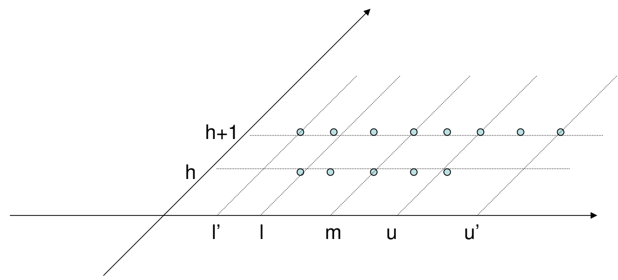

Figure 1: Schematic representation of two qukit strings on a 2 dimensional integer lattice. The strings at h and $h+1$ with 5 and 8 qukits extend from $j = l$ to $j = u$ and from $j = l'$ to $j = u'$ respectively. The site $j = m$ is occupied by both a qukit and the sign qubit. It is also the location of the $k - al$ point. The equal distance between adjacent qukits shown here is for illustrative purposes only. No metric distance between adjacent points is assumed here.

expressed here as $63 - 71,\ 459+,\ 0 - 0753$.

To this end let l, m, u be integers where $l \leq m \leq u$. Define the q_k string state $|\gamma, (m, h), s, l, u\rangle_k$ on the integer pair interval $[(l, h), (u, h)]$ by

$$\begin{aligned} |\gamma, (m, h), s, l, u\rangle_k &= c^\dagger_{\gamma,(m,h)}(a^\dagger_k)_{s(u,h),(u,h)} \cdots (a^\dagger_k)_{s(l,h),(l,h)}|0\rangle \\ &= c^\dagger_{\gamma,m,h}(a^\dagger_k)^s_{[l,u]}|0\rangle. \end{aligned} \tag{1}$$

Here s is a $0, 1, \cdots, k-1$ valued function on $[(l, h), (u, h)]$, $\gamma = +, -$, and (m, h) is the location of the sign and the "k-al" point. The pair (m, h) also serves as a useful denotation of a string location where the string extends from (l, h) to (u, h). In the above $c^\dagger_{\gamma,m,h}(a^\dagger_k)^s_{[l,u]}$ is a shorthand notation for the creation operator string in the definition.

This representation is sufficiently broad to encompass all three types of numbers. It also includes qukit string states with leading and trailing $0s$. For integers, I, and natural numbers, N, s satisfies the requirement that $s(j, h) = 0$ for all $j : l \leq j < m$. For N, $\gamma = +$. For rational number representations there are no restrictions on s or γ.

The states $|\gamma, (m, h), s, l, u\rangle_k$ and their linear superpositions can be regarded as elements of a Fock space, $\mathcal{F}^X_{k,(m,h)}$, that is spanned by a basis set $\mathcal{B}^X_{k,(m,h)}$ of states $|\gamma, (m, h), s, l, u\rangle_k$. Here $X = N, I,$ or Ra, m and h are fixed, l and u can vary with $l \leq m \leq u$, and, for $X = Ra$, s is any $0, \cdots k-1$ valued function on the interval $[(l, h), (l+1, h), \cdots, (u, h)]$. For $X = I$ and N, s satisfies the restriction given above. Note that the states $|\gamma, (m, h), s, l, u\rangle_k$ are pairwise orthogonal,

$$_k\langle\gamma', (m', h')s', l', u'|\gamma, (m, h), s, l, u\rangle_k = \delta_{\gamma',\gamma}\delta_{s',s}\delta_{(m'h'),(m,h)}\delta_{(l',u'),(l,u)}. \tag{2}$$

The Fock space representation is used because linear superpositions of states with different numbers of q_k are included.

In the following the action of operators on pairs and triples, and more generally $n-tuples$ of these states needs to be considered. States representing pairs or $n-tuples$ of q_k strings, at different h values can either be represented by extension of the Fock space representation or by considering tensor products of the Fock spaces $\mathcal{F}^X_{k,(m,h)}$,. In either case the basic units are the basis states given by Eq. 1 where $c^\dagger_{\gamma,m,h}(a^\dagger_k)^s_{[l,u]}$ acts like a q_k string state creation operator. (This description is assumed to also include the sign qubit implicitly.)

The Fock space extension for pairs of states of q_k strings, one at (m',h') and the other at (m,h), is represented by

$$|\gamma',(m',h'),s',l',u';\gamma,(m,h),s,l,u\rangle_k = c^\dagger_{m',h'}(a^\dagger_k)^{s'}_{[l',u']}c^\dagger_{m,h}(a^\dagger_k)^s_{[l,u]}|0\rangle. \tag{3}$$

Here the only restriction is that the two strings do not overlap for either bosons or fermions. This is accounted for by requiring that $h \neq h'$. There is no restriction on the values of m and m'.

The other representation, which is the one which will be used here, is to represent pairs of q_k string states by $|\gamma',(m',h'),s',l',u'\rangle_k|\gamma,(m,h),s,l,u\rangle_k$. In general, $n-tuples$ have a similar representation. These states and their linear superpositions are elements of tensor products of the spaces $\mathcal{F}^X_{k,(m,h)}$.

The correspondence between the two representations is given by

$$|\gamma',(m',h'),s'l',u';\gamma,(m,h),s,l,u\rangle_k \Leftrightarrow |\gamma',(m',h'),s'l',u'\rangle_k|\gamma,(m,h),s,l,u\rangle_k \tag{4}$$

for pairs. For $n-tuples$ the correspondence is given by

$$\begin{aligned}&|\gamma_1,(m_1,h_1),s_1,l_1,u_1;\cdots;\gamma_n,(m_n,h_n),s_n,l_n,u_n\rangle_k \\ &\quad = |\gamma_1,(m_1,h_1),s_1,l_1,u_1\rangle_k\cdots|\gamma_n,(m_n,h_n),s_n,l_n,u_n\rangle_k.\end{aligned} \tag{5}$$

From now on the product state representation will be used.

The basic arithmetic relations and operations consist of equality $=$, and an ordering for all three number types, and addition $+$ and multiplication, $\times$, for the natural numbers. For the integers, subtraction $-$, is added, and for rational numbers, division $\div$, is added. The properties of these relations and operations are given by the three sets of axioms for natural numbers, N,[3] integers, I, and rational numbers, Ra.

There are two ways to approach the problem of showing that the states $|\gamma,(m,h),s,l,u\rangle_k$ represent numbers. One way is to define an operator $\tilde{N}_k$ whose eigenstates are the states $|\gamma,(m,h),s,l,u\rangle_k$ and whose eigenvalues are the N, I, or Ra equivalent numbers in the real number base R of the Fock spaces. One knows that the sets of eigenvalues of $\tilde{N}_k$ as subsets of R satisfy the relevant axioms for the arithmetic relations and operations on the relevant subsets of R. This can be used to require that the arithmetic relations and operations on the states $|\gamma,(m,h),s,l,u\rangle_k$ be defined so that the operator $\tilde{N}_k$ is an arithmetic isomorphism. That is, it preserves the basic arithmetic relations

[3] For N a successor operation should also be included. This operation, and its use to define the other arithmetic operations, is discussed in [20]

and operations. If this is true, then it follows that the states represent numbers as they satisfy the relevant axioms.

An example of an operator for this role is given by the definition of $\tilde{N}_k$ as the product of two commuting operators, a sign scale operator $(\tilde{N}_k)_{ss}$, and a value operator $(\tilde{N}_k)_v$. One has

$$\begin{array}{c}\tilde{N}_k = (\tilde{N}_k)_{ss}(\tilde{N}_k)_v \\ \text{where } (\tilde{N}_k)_{ss} = \sum_{\gamma,m} \gamma k^{-m} c^\dagger_{\gamma,m} c_{\gamma,m} \\ \tilde{N}_v = \sum_{\alpha,j,h} \alpha k^j (a^\dagger_k)_{\alpha,(j,h)} (a_k)_{\alpha,(j,h)}.\end{array} \tag{6}$$

Note that, because of the presence of strings of leading or trailing $0s$, the eigenspaces of $\tilde{N}$ are infinite dimensional. The eigenspace for the number 0 includes the state $|0+\rangle = c^\dagger_{+,m}(a^\dagger_k)_{0,m}|0\rangle$ and all states of the form $|\gamma,(m,h),s,l,u\rangle_k$ where s is the constant 0 function $\bar{0}_{[l,u]}$ of $0s$ from l to u.

The other way to show that the states $|\gamma,(m,h),s,l,u\rangle_k$ represent numbers is to define basic arithmetic relations and operations in terms of simple operations on the states and show that the definitions given do satisfy the relevant axioms. The main advantage of this approach is that it is more direct in that it makes no use of elements of R and their arithmetic properties. This more direct approach is the one used elsewhere [20, 21] and is followed here. However, here the definitions in terms of simple operations are included by reference so as not to repeat other work. In any case one notes that the assertion that the states $|\gamma,(m,h),s,l,u\rangle_k$ represent numbers is relative to the definitions of the arithmetic relations and operations and their satisfaction of relevant axioms. It has no absolute or intrinsic meaning.

In the following the values of l,u will often be suppressed in state representations. In this case the state $|\gamma,(m,h),s,l,u\rangle_k$ will be represented in a short form as $|\gamma,(m,h),s\rangle_k$, where the values of l,u are included in the definition of s.

The basic relations and operations are defined for each value of k and for different values of (m,h). Arithmetic equality, $=_{A,k,m,h,h'}$, as a binary relation, is defined between states in $\mathcal{F}^X_{k,m,h}$ and $\mathcal{F}^X_{k,m,h'}$ by

$$\begin{array}{c}|\gamma,(m,h),s\rangle_k =_{A,k,m,h,h'} |\gamma',(m,h'),s'\rangle_k, \\ \text{if } \gamma'=\gamma \text{ and for all } j \\ \text{If } j \text{ is in both } [l,u] \text{ and } [l',u'], \text{ then } s(j,h)=s'(j,h'), \\ \text{If } j \text{ is in } [l,u] \text{ and not in } [l',u'], \text{ then } s(j,h)=0. \\ \text{If } j \text{ is in } [l',u'] \text{ and not in } [l,u], \text{ then } s'(j,h')=0.\end{array} \tag{7}$$

Here $X = N, I$, or Ra. This definition is complex because it defines equality up to leading and trailing $0s$. The extension of the definition to different m values as in $=_{A,k,m,h,m',h'}$ is more complex in that the difference between m and m' must be taken into account.

Note that quantum state equality implies arithmetic equality:

$$(|\gamma,(m,h),s\rangle_k = |\gamma',(m,h'),s'\rangle)_k \to (|\gamma,(m,h),s\rangle_k =_{A,k,m,h,h'} |\gamma',(m,h'),s'\rangle_k).$$

However the converse implication does not hold.

Arithmetic ordering $\leq_{A,k,m,h,h'}$ on N, and on positive I and Ra states is defined by

$$\begin{array}{l} |+,(m,h),s\rangle_k \leq_{A,k,m,h,h'} |+,(m,h'),s'\rangle_k \\ \quad \leftrightarrow \left(\begin{array}{l} |+,(m,h),s\rangle_k =_{A,k,m,h,h'} |+,(m,h'),s'\rangle_k \text{ or} \\ |+,(m,h),s\rangle_k <_{A,k,m,h,h'} |+,(m,h'),s'\rangle_k \end{array} \right. \end{array} \tag{8}$$

where

$$\begin{array}{l} |+,(m,h),s\rangle_k <_{A,k,m,h,h'} |+,(m,h'),s'\rangle_k \text{ if} \\ \quad \text{for some } j \text{ in both } [l,u] \text{ and } [l',u'], \\ \quad\quad s(j,h) < s'(j,h') \text{ and } s_{[(j+1,h),(u,h)]} = s'_{[(j+1,h'),(u',h')]} \\ \quad\quad\quad \text{up to leading } 0s. \end{array} \tag{9}$$

The extension to zero and negative I and Ra states is given by

$$\begin{array}{c} |+,(m,h),0\rangle_k \leq_{A,k,m,h,h'} |+,(m,h'),s'\rangle_k \text{ for all } s' \\ |+,(m,h),s\rangle_k \leq_{A,k,m,h,h'} |+,(m,h'),s'\rangle_k \\ \rightarrow |-,(m,h'),s'\rangle_k \leq_{A,k,mh.h'} |-,(m,h),s\rangle_k. \end{array} \tag{10}$$

The definitions of the arithmetic relations, $=_{A,k,m,h,h'}, \leq_{A,k,m,h,h'}$ are given for specific values of $m, h,$ and h'. These restrictions can be removed by extending the definitions to apply also to arbitrary values of (m,h) and (m',h'). The only restriction is that any pair of strings being compared have no overlap. This is the case if and only if $h \neq h'$. This is the case because the values of h distinguish the different strings whereas the values of m locate the sign and "$k = al$" point in a string.

In the following it is often useful to define arithmetic relations that are equivalent to sums over all pairs of $h, \neq h'$. That is

$$\begin{array}{l} =_{A,k,m} \leftrightarrow \exists h,h' =_{A,k,m,h,h'} \\ \leq_{A,k,m} \leftrightarrow \exists h,h' \leq_{A,k,m,h,h'} . \end{array} \tag{11}$$

Projection operators $\tilde{P}_{=_{A,k,m}}$ and $\tilde{P}_{\leq_{A,k,m}}$ can be associated with these relations. The action of $\tilde{P}_{=_{A,k,m}}$ on pairs of basis states for arbitrary $h, h' \neq h$ is given by

$$\begin{array}{l} \tilde{P}_{=_{A,k,m}} |\gamma,(m,h),s''\rangle_k |\gamma',(m,h'),s'\rangle_k = \\ \left\{ \begin{array}{l} |\gamma,(m,h),s''\rangle_k |\gamma',(m,h'),s'\rangle_k \\ \qquad \text{if } |\gamma,(m,h),s''\rangle_k =_{A,k,m} |\gamma',(m,h'),s'\rangle_k, \\ 0 \text{ if } |\gamma,(m,h),s''\rangle_k \neq_{A,k,m} |\gamma',(m,h'),s'\rangle_k. \end{array} \right. \end{array} \tag{12}$$

$\tilde{P}_{=_{A,k,m}}$ can also be written as

$$\tilde{P}_{=_{A,k,m}} = \sum_{\gamma, s^{diff}} \sum_{h'>h} \tilde{P}_{\gamma,[s],k,h} \tilde{P}_{\gamma,[s],k,h'} \tag{13}$$

where

$$\tilde{P}_{\gamma,[s],k,h} = \sum_{s'\sim_0 s} \tilde{P}_{|\gamma,(m,h),s',l,u\rangle_k}. \tag{14}$$

In terms of A-C operators $\tilde{P}_{|\gamma,(m,h),s',l,u\rangle_k}$ can be expressed as

$$\tilde{P}_{|\gamma,(m,h),s',l,u\rangle_k} = c^\dagger_{\gamma,m,h}(a^\dagger_k)^s_{[l,u]}(a_k)^s_{[l,u]}c_{\gamma,m,h}. \tag{15}$$

Here $c^\dagger_{\gamma,m,h}(a^\dagger_k)^s_{[l,u]}$ is given by Eq. 1. In these equations the sum over s^{diff} is restricted to those s that have no leading or trailing 0s. The sum over $s' \sim_0 s$ is over all s' that differ from s only by leading or trailing 0s.

This definition can be applied to linear superposition states. The probability that $\psi =_{A,k,m} \phi$ where $\psi = \sum_{\gamma,s} c_{\gamma,s}|\gamma,(m,h),s\rangle_k$ and $\phi = \sum_{\gamma,s} d_{\gamma,s}|\gamma,(m,h'),s\rangle_k$ is given by

$$\langle\psi\phi|\tilde{P}_{=_{A,k,m}}|\psi\phi\rangle = \sum_{\gamma,s^{diff}} \sum_{s'\sim_0 s} \sum_{s''\sim_0 s} |c_{\gamma,s'}|^2|d_{\gamma,s''}|^2 \tag{16}$$

As a simple example let $\psi = 1/\sqrt{2}(|22+\rangle + |022+\rangle)$ and $\phi = 1/\sqrt{2}(|22+0\rangle + |121+\rangle)$. The probability that these two states are arithmetically equal is 1/2 even though they are quantum mechanically orthogonal.

An equation similar to Eq. 12 holds for $\tilde{P}_{\leq_{A,k,m}}$ where

$$\tilde{P}_{\leq_{A,k,m}} = \sum_{\gamma,s,\gamma',s':\ \gamma,s\leq\gamma',s'} \tilde{P}_{[\gamma,s]} \times \tilde{P}_{[\gamma',s']}. \tag{17}$$

Here $\gamma, s \leq \gamma', s'$ is defined by Eqs. 8-10.

The basic arithmetic operations are $+$, $-$, and $\times$. Division will be considered later. For each k and m, unitary operators for $+$, $-$, and $\times$, are represented by $\tilde{+}_{A,k,m}$, $\tilde{-}_{A,k,m}$, and $\tilde{\times}_{A,k,m}$. These operators act on pairs of q_k string states as input. Output consists of the pair of input states and a result string states. To express this in more detail, let $\tilde{O}_{A,k,m}$ represent any of the three operations, ($O = +, -$, or $O = \times$.) Then

$$\begin{aligned}&\tilde{O}_{A,k,m}|\gamma,(m,h),s\rangle_k|\gamma',(m,h'),s'\rangle_k \\ &\quad = |\gamma,(m,h),s\rangle_k|\gamma',(m,h'),s'\rangle_k|\gamma'',(m,h''),s''\rangle_{k,O}\end{aligned} \tag{18}$$

The preservation of the input states is sufficient to ensure that the operators are unitary. The values of h, h', h'' are arbitrary except that they are all different.

In these equations the states $|\gamma'',(m,h''),s''\rangle_k$ with subscripts $O = +, -, \times$ give the results of the arithmetic operations. It is often useful to write them as

$$\begin{aligned}|\gamma'',(m,h''),s''\rangle_{k,+} &= |(m,h''),(\gamma',s' +_A \gamma,s)\rangle_k, \\ |\gamma'',(m,h''),s''\rangle_{k,-} &= |(m,h''),(\gamma',s' -_A \gamma,s)\rangle_k, \\ |\gamma'',(m,h''),s''\rangle_{k,\times} &= |(m,h),(\gamma',s' \times_A \gamma,s)\rangle_k.\end{aligned} \tag{19}$$

The subscript A on these operations distinguishes them as arithmetic operations. They are different from the quantum operations of linear superposition, $+, -$ and product, $\times$ with no subscripts.

The action of these linear operators on general states $\phi = \sum_{\gamma,s} c_{\gamma,s}|\gamma,(m,h),s\rangle_k$, $\psi = \sum_{\gamma',s'} d_{\gamma',s'}|\gamma',(m,h'),s'\rangle_k$ is given by

$$\begin{array}{c}\tilde{O}_{A,k,m}\phi\psi = \sum_{\gamma,s}\sum_{\gamma',s'} c_{\gamma,s}d_{\gamma',s'}|\gamma,(,m,h),s\rangle_k \\ \times|\gamma',(m,h'),s'\rangle_k|(,m,h''),(\gamma,sO_A\gamma',s')\rangle_k.\end{array} \tag{20}$$

For linear superposition states arithmetic properties, including those expressed by the axioms and theorems, are true with some probability between 0 and 1. For example the probability that $\phi =_{A,k,m} \psi$ is true is given by $(\phi\psi|\tilde{P}_{=_{A,k,m}}|\phi\psi)$ with $\tilde{P}_{=_{A,k,m}}$ given by Eq. 12. A similar expression holds with $\leq_{A,k,m}$ replacing $=_{A,k,m}$.

Properties involving the arithmetic operations have complex expressions because the operations induce entanglement. For example, the probability that the axiom, $x+y = y+x$, that expresses the commutativity of addition, is true is obtained as follows: One needs to compare the result of the addition $\tilde{+}_{A,k,m}\phi\psi$ with the result of the addition $\tilde{+}_{A,k,m}\psi\phi$. These two results are obtained by carrying out the additions and taking the traces over both input components. For ϕ and ψ the results are given by two density operators(m and h are suppressed in the states):

$$\begin{array}{l}\rho_{\phi+_{A,k}\psi} = Tr_{|\gamma,s\rangle_k,|\gamma',s'\rangle_k}(\tilde{+}_{A,k,m}\phi,\psi) \\ = \sum_{\gamma,s}\sum_{\gamma',s'}|c_{\gamma,s}|^2|d_{\gamma',s'}|^2|\gamma,s+_A\gamma',s'\rangle_k\langle\gamma,s+_A\gamma',s'| \\ \rho_{\psi+_{A,k}\phi} = Tr_{|\gamma',s'\rangle_k,|\gamma,s\rangle_k}(\tilde{+}_{A,k,m}\psi,\phi) = \\ \sum_{\gamma,s}\sum_{\gamma',s'}|d_{\gamma',s'}|^2|c_{\gamma,s}|^2|\gamma',s'+_{A,k}\gamma,s\rangle_k\langle\gamma,s+_{A,k}\gamma',s'|.\end{array} \tag{21}$$

For these two density operators the probability that

$$\rho_{\phi+_{A,k}\psi} =_{A,k} \rho_{\psi+_{A,k}\phi} \tag{22}$$

is true is given by

$$Tr(\tilde{P}_{=_{A,k,m}}\rho_{\phi+_{A,k}\psi}\times\rho_{\psi+_{A,k}\phi}) \tag{23}$$

with $\tilde{P}_{=_{A,k,m}}$ given by Eq. 12. The trace is taken over all remaining variables. Note that in evaluating this by carrying out the sums involved, one uses the commutativity of addition for basis states, $|\gamma,s+_{A,k}\gamma',s'\rangle_k =_{A,k,m} |\gamma',s'+_{A,k}\gamma,s\rangle_k$.

The division operator $\tilde{\div}_{A,k,m}$, which applies to the rational numbers only, satisfies

$$\begin{array}{l}\tilde{\div}_{A,k,m}|\gamma,(m,h),s\rangle_k|\gamma',(m,h'),s'\rangle_k \\ \quad = |\gamma,(m,h),s\rangle_k|\gamma',(m,h'),s'\rangle_k|(m,h''),(\gamma,s\div_A\gamma',s')\rangle_{k'} \\ \qquad \text{where } k' = k'(k,|\gamma',(m,h'),s'\rangle).\end{array} \tag{24}$$

This definition has the advantage that it satisfies the usual axioms of division, such as closure under division. However it has the problem that the quotient state $|(m,h''),(\gamma',s'\div_A\gamma,s)\rangle_{k'}$ resulting from dividing $|\gamma',s'\rangle_k$ by $|\gamma,s\rangle_k$ is, in many cases, not a base k state. Instead it is a base k' state where k' is a function

of both k and the state $|\gamma, (m,h), s\rangle_k$. In addition, this representation of division, applied to linear superposition states as divisors, results in base entanglement.

Examples are the best way to see this. The base 10 inverse of 13 is given by,

$$\tilde{\div}|1+0\rangle_{10}|13+\rangle_{10} = |1+0\rangle_{10}|13+\rangle_{10}|0+(01)\rangle_{13} \tag{25}$$

As shown the result corresponds to the base 13 number 0.1 or as 0.(01). The second example corresponds to the base 10 division $1740 \div 13$

$$\begin{aligned}&\tilde{\div}|(1)(7)(4)(0)+\rangle_{10}|(1)(3)+\rangle_{10}\\ &\qquad = |(1)(7)(4)(0)+\rangle_{10}|(1)(3)+\rangle_{10}|(10)(03)+(11)\rangle_{13}.\end{aligned} \tag{26}$$

The base 13 answer corresponds to 0.1×1740 converted to base 13 which is (10)(03).(11).

This use of decimal numbers in parentheses is a convenient way to denote base k digits. One needs k distinct digits $0, 1, \cdots, k-1$. The idea is to let numbers in parentheses denote the digits. Thus (31) denotes the $32nd$ digit for any base $k \geq 32$, etc.

A simple example to show both base and state entanglement is the division of $|(1)(7)(4)(0)+\rangle_{10}$ by the state $1/\sqrt{2}(|(1)(3)+\rangle_{10} + |(2)(1)+\rangle_{10})$. The result, given by

$$\begin{aligned}&\tilde{\div}_{A,10}|(1)(7)(4)(0)+\rangle_{10}1/\sqrt{2}(|(1)(3)+\rangle_{10} + |(2)(1)+\rangle_{10}) = |(1)(7)(4)(0)+\rangle_{10}\\ &\times 1/\sqrt{2}(|(1)(3)+\rangle_{10}|(10)(03)+(11)\rangle_{13} + |(2)(1)\rangle_{10}|(03)(19)+(14)\rangle_{21}),\end{aligned} \tag{27}$$

shows entanglement of base 10 and 13 states with base 10 and 21 states. The base entanglement can be removed by converting both $|(10)(03)+(11)\rangle_{13}$ and $|(03)(19)+(14)\rangle_{21}$ to base $13 \times 21 = 273$ base states. The state entanglement with the divisor component states still remains as does the base change.

There are two ways to handle this problem. One is to use this definition and deal with the complexity of base changes. The advantages are that, with this definition, the axioms for Ra for division are simple in that Ra is closed under division.

The other option is to use a definition of division to arbitrary accuracy. This definition, which is much used in actual computations by computers, relies on the fact that one can always stop the generation of an infinite string of digits at any point. This is easiest to see in computing inverses. For example 1/3 in the decimal base is an infinite string of $3s$ to the right of the decimal point. Limiting the string to ℓ $3s$ to the right of the decimal point is division to accuracy ℓ as the result is accurate to $10^{-\ell}$.

This definition will be used here because there is no change of base. This greatly simplifies the treatment. The difficulty is the complexity of the axioms that express the concept of "division to arbitrary accuracy". However since there is no emphasis on axiomatic details here, this is not much of a problem.

The definition of an operator $\tilde{\div}_{A,k,m,\ell}$ for each ℓ is similar to that for $\tilde{O}_{A,k,m}$ in Eq. 18:

$$\begin{aligned}&\tilde{\div}_{A,k,m,\ell}|\gamma, (m,h), s\rangle_k|\gamma', (m,h), s'\rangle_k\\ &\qquad = |\gamma, (m,h), s\rangle_k|\gamma', (m,h'), s'\rangle_k|\gamma'', (m,h''), s''\rangle_{k,\div_\ell}.\end{aligned} \tag{28}$$

The quotient state

$$|\gamma'', (m, h''), s'', l'', u''\rangle_{k,\div_\ell} = |(m, h'')(\gamma, s \div_{A,\ell} \gamma', s'\rangle_k \tag{29}$$

satisfies the condition,

$$\text{If } l'' < m - \ell, \text{ then } s''(j, h'') = 0 \text{ for all } j < m - \ell.$$

In other words, the quotient state is accurate to $|+, (m, h''), 0_{[m-\ell+1,m]} 1_{m-\ell}\rangle_k$. Since the $\tilde{N}$ eigenvalue of this state is $k^{-\ell}$, this is equivalent to saying that the result is accurate to $k^{-\ell}$. More details on $\tilde{\div}_{A,k,m,\ell}$ are given in [21, 23].

3 Space of Quantum Theory Representations of Numbers

So far a quantum representation of numbers as states of finite strings of base k qukits q_k has been described. The states are elements of a basis set $\mathcal{B}_{k,(m,h)}$ that spans a Fock space $\mathcal{F}^X_{k,(m,h)}$ where $X = N, I$, or Ra. Arithmetic relations and operations are, in general, defined on $n - tuples$ of q_k string states. These states are elements of $n - fold$ tensor products of $\mathcal{F}^X_{k,(m,h)}$. If $S = \{(m, h)\}$ is a finite subset of n pairs of integers where the values of h are all different then an $n - tuple$ operation or relation would be defined on $\mathcal{F}^X_{k,S}$ where

$$\mathcal{F}^X_{k,S} = \bigotimes_{(m,h)\epsilon S} \mathcal{F}^X_{k,(m,h)}. \tag{30}$$

If the definitions of the relations and operations are insensitive to the values of (m, h), then the appropriate domain of definition would be the space $\mathcal{F}^X_{k,n} = \bigoplus_{S:|S|=n} \mathcal{F}^X_{k,S}$ or the space $\mathcal{F}^X_k = \bigoplus_S \mathcal{F}^X_{k,S}$. Here the sum is over all finite subsets S of $I \times I$.

From this one sees that $\mathcal{F}^X_{k,(m,h)}$ is the basic space as it contains states of single strings of q_k. These states and the space, $\mathcal{F}^X_{k,(m,h)}$, are parameterized by three parameters, k, m, h that represent the number base, and the location of a string in $I \times I$. m is the location of the sign and "$k - al$" point in a string and h is the location of a string.[4]

There remains a degree of freedom that should be accounted for. This is the freedom of gauge fixing or basis choice for the states of each q_k. This is represented here by a gauge fixing function, g, that chooses a basis set for the k dimensional Hilbert space of states for each integer pair in $I \times I$. That is, $g(k, j, h)$ is the basis set of the states of q_k that span the k dimensional Hilbert state space $\mathcal{H}^k_{j,h}$ at site j, h.

[4] Note that the two dimensions in $I \times I$ are treated differently. This is a consequence of the fact that one cannot have product states of two q_k strings with the same value of h but different values of m. One does not want overlapping strings of $q'_k s$ with the same k because one does not know which q_k belong to which string in the overlap regions. This problem does not exist for overlapping q_k and $q_{k'}$ strings where $k' \neq k$. Such overlaps are allowed.

The choice of a gauge or basis set for each qukit at each location is often referred to as a choice of quantization axis at each site (j, h). Physically this is represented by a vector field on $I \times I$, such as a magnetic field. It is also described as a moving frame [24]. As defined g is a function from $N \geq 2 \times I \times I \times$ to basis sets of finite dimensional Hilbert spaces where $g(k, j, h)$ is a basis for $\mathcal{H}^k_{j,h}$.

The inclusion of g as a component of the parameter space is done because here it is an independent variable. No external fields are present to determine g. The function g is also quite different from the other components of the space. Unlike the other components, it is not a property of the q_k or q_k strings. As a k dependent choice of a basis set for each site,[5] it is essential for the description of arithmetic properties of the q_k strings. This was seen in the previous section where the descriptions of arithmetic relations and operations are given relative to a basis set of states of qukit strings.

These considerations indicate that a subscript g should be added to all the basis states and arithmetic relations and operations discussed in Section 2. It was not done there to conform with general usage. However it will be included where appropriate from now on.

The next step is to associate a state space and basis with each point of the parameter space. Here this association is defined by the three maps

$$(k, (m, h), g) \rightarrow (\mathcal{F}^X_{k,(m,h)}, \mathcal{B}_{k,(m,h),g}) \tag{31}$$

one each for $X = N, I, Ra$. The righthand site contains two elements, a Fock space $\mathcal{F}^X_{k,(m,h)}$ and a set $\mathcal{B}_{k,(m,h),g}$. This set is the product of all basis sets $\{g(k, j, h) : j \epsilon I\}$. It also includes a basis for the sign qubit at (m, h). The elements of $\mathcal{B}_{k,(m,h),g}$ are the qukit string states $|\gamma, (m, h), s, l, u\rangle_{k,g}$ for all values of γ, s, l, u.

To save on notation the pair $(\mathcal{F}^X_{k,(m,h)}, \mathcal{B}_{k,(m,h),g})$ will be denoted by $\mathcal{FB}^X_{k,(m,h),g}$. The set of all $\mathcal{FB}^X_{k,(m,h),g}$ for all values of the parameters in the parameter set is the space of representations referred to in the title of this paper.

3.1 Transformations

So far one has a representation space with elements $\mathcal{FB}^X_{k,(m,h),g}$ of the space parameterized by the $4 - tuples$ $((m, h), k, g)$. It is of interest to investigate transformations on the parameter set and their correspondents on the representation space.

To this end one notes that transformations $(k, (m, h), g) \rightarrow (k', (m', h'), g')$ in the parameter space induce transformations $\mathcal{FB}^X_{k,(m,h),g} \rightarrow \mathcal{FB}^X_{k',(m',h'),g'}$ in the representation space. This is shown schematically in Fig. 2. The transformations are of three types, translations, base changes, and gauge transformations.

[5] The k dependence of g can also be represented as a subscript as in $g_k(j, h)$. In this case there are many g functions, one for each k.

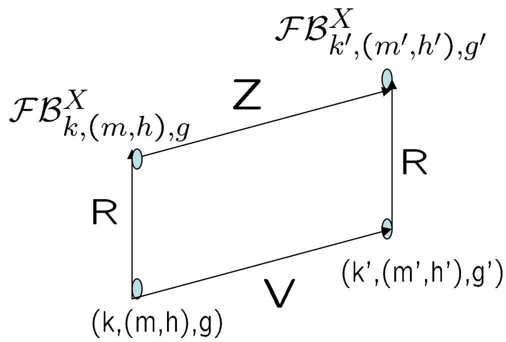

Figure 2: Schematic Diagram of the Commutation Condition for Parameter Set and Induced Representation Space Transformations. If $V((k,(m,h),g)) = (k',(m',h'),g')$, and $R((k,(m,h),g)) = \mathcal{FB}^X_{k,(m,h),g}$, and $Z(\mathcal{FB}^X_{k,(m,h),g}) = \mathcal{FB}^X_{k',(m',h'),g'}$, then V, R, Z must satisfy $R(V((k,(m,h),g))) = Z(R((k,(m,h),g)))$.

3.2 Translations and Base Changes

There are two translation operators, $\tilde{T}_1$ and $\tilde{T}_2$. These operators shift the q_k string one step by shifting either the first or the second integer parameter by $+1$. One has

$$\begin{aligned}\tilde{T}_1|\gamma,(m,h),s,(l,h),(u,h)\rangle_{k,g} &= |\gamma,(m+1,h),s',(l+1,h),(u+1,h)\rangle_{k,g}\\ \tilde{T}_2|\gamma,(m,h),s,(l,h),(u,h)\rangle_{k,g} &= |\gamma,(m,h+1),s,(l,h+1),(u,h+1)\rangle_{k,g}\end{aligned} \tag{32}$$

where $s'(j+1,h) = s(j,h)$ for $l \geq j \geq u$ and s, s' are independent of h. $\tilde{T}_1$ and $\tilde{T}_2$ can be regarded as step translation operators in the direction along the q_k string or at right angles to the string.

$\tilde{T}_1$ and $\tilde{T}_2$ are unitary. They also have the property that for each k and (m,h)

$$\begin{aligned}\tilde{T}_1\mathcal{FB}^X_{k,(m,h),g} &= \mathcal{FB}^X_{k,(m+1,h),g}\\ \tilde{T}_2\mathcal{FB}^X_{k,(m,h),g} &= \mathcal{FB}^X_{k,(m,h+1),g}.\end{aligned} \tag{33}$$

Also the states $\tilde{T}_1|\gamma,m,s,l,u\rangle_{k,g}$ and $\tilde{T}_2|\gamma,m,s,l,u\rangle_{k,g}$ represent the same number in $\mathcal{FB}^X_{k,(m+1,h),g}$ and in $\mathcal{FB}^X_{k,(m,h+1),g}$ as $|\gamma,m,s,l,u\rangle_{k,g}$ does in $\mathcal{F}^{XB}_{k,(m,h),g}$.

Transformations that change bases are more complex, especially for rational number states. For each pair k, k' of bases, let $\tilde{W}_{k',k}$ denote the operator that changes any state from base k to one in base k'. The operator must satisfy the requirement that for each state $|\gamma,(m,h),s\rangle_{k,g}$ in $\mathcal{FB}^X_{k,(m,h),g}$ on which $\tilde{W}_{k',k}$ is defined, $\tilde{W}_{k',k}|\gamma,(m,h),s\rangle_{k,g}$, as a state in $\mathcal{FB}^X_{k',(m,h),g}$, represents the same number as $|\gamma,(m,h),s\rangle_{k,g}$ does in $\mathcal{FB}^X_{k,(m,h),g}$. Note that if $\tilde{W}_{k',k}$ is defined for

one value of (m,h) it is defined for all values of (m,h). For notational simplicity, l and u have been suppressed.

It is worth giving two examples of base changes. The symbol representation used for the division examples, Eqs. 26, 27, is used here. The base 37 integer $(23)(35)(0)._{37}$, also represented as $23\times 37^2+35\times 37+0$, has a base 10 representation as $(3)(2)(7)(8)(2)._{10}$. As another example, the rational number $(23)(35)(0).(1)_{37}$, where the subscript denotes the base 37, has an equivalent representation as $23\times 37^2+35\times 37+0+1/37$. This number does not have a finite string representation in base $k=10$.

As a transformation operator on the representation space $\tilde{W}_{k',k}$ is very different from $\tilde{T}_1$ and $\tilde{T}_2$ in that these operators, and the to-be-described gauge changing operator, are defined on states of individual q_k. $\tilde{W}_{k',k}$ is defined on states in $\mathcal{FB}^X_{k,(m,h)}$ as it is a string state changing operator. It is not defined on states of individual q_k. This will become quite evident in the following.

The requirement that $\tilde{W}_{k',k}|\gamma,m,s,l,u\rangle_{k,g}=|\gamma,m,s',l',u'\rangle_{k',g}$ represent the same number as does $|\gamma,m,s,l,u\rangle_k$ is not trivial. It means that both states must have exactly the same numerical properties, expressed as theorems derivable from the set of axioms for the number type being considered. A rigorous proof of this would require proving that the axiomatic properties of the basic arithmetic relations and operations are preserved, and that the logical deduction rules for obtaining theorems from axioms are invariant. A necessary condition for this to be true is the requirement that the properties of the basic arithmetic relations and operations, as given by the axioms for the number type being considered, are preserved.

For example, for $X=I$ the basic arithmetic relations and operations are $=_{A,k,m,g},\leq_{A,k,m,g},+_{A,k,m,g},\times_{A,k,m,g}$. The requirement that $\tilde{W}_{k',k}$ preserves these properties for the relations is expressed by (h is suppressed here)

$$\begin{array}{l}|\gamma,m,s,l,u\rangle_{k,g}=_{A,k,m,g}|\gamma',m,s',l,u'\rangle_{k,g}\\ \rightarrow\tilde{W}_{k',k}|\gamma,m,s,l,u\rangle_{k,g}=_{A,k',m,g}\tilde{W}_{k',k}|\gamma',m,s',l,u'\rangle_{k,g},\\ |\gamma,m,s,l,u\rangle_{k,g}\leq_{A,k,m,g}|\gamma',m,s',l,u'\rangle_{k,g}\\ \rightarrow\tilde{W}_{k',k}|\gamma,m,s,l,u\rangle_{k,g}\leq_{A,k',m,g}\tilde{W}_{k',k}|\gamma',m,s',l,u'\rangle_{k,g}\end{array}\tag{34}$$

. The requirement for the arithmetic operations $+_{A,k,m,g},\times_{A,k,m,g}$ and $-_{A,k,m,g}$ can be expressed by

$$\begin{array}{l}\tilde{W}_{k',k}|(\gamma,m,s,l,u)O_{A,k,m,g}(\gamma',m,s',l',u')\rangle_{k,g}\\ \qquad=_{A,k'm,g}|(W_{k',k}(\gamma,m,s,l,u))O_{A,k',m,g}(W_{k',k}(\gamma',m,s',l',u)')\rangle_{k',g}\end{array}\tag{35}$$

where O stands for $+$, $\times$ and $-$. This equation says that transforming the state resulting from carrying out the operation $O_{A,k,m,g}$ gives a state that is arithmetically equal to the state resulting from carrying out $O_{A,k',m,g}$ on the transformed states. The states appearing in the above must satisfy the restrictions on s,s' for integer states: If $l\leq j<m$, then $s(j)=0$ and $s'(j)=0$.

The operator $\tilde{W}_{k',k}$ has different properties for $X=Ra$ than for $X=N,I$. For $X=N,I$, $\tilde{W}_{k',k}$ is defined on all of $\mathcal{FB}^X_{k,(m,h),g}$ and is unitary. In this case,

for all k',
$$\tilde{W}_{k',k}\mathcal{FB}^X_{k,(m,h),g} = \mathcal{FB}^X_{k',(m,h),g}. \tag{36}$$
Also $\tilde{W}_{k,k}$ is the identity on $\mathcal{FB}^X_{k,(m,h),g}$, and $\tilde{W}^\dagger_{k',k} = \tilde{W}_{k,k'}$. Also the operators $\tilde{W}_{k',k}$ have the group multiplication property in that
$$\tilde{W}_{k'',k} = \tilde{W}_{k'',k'}\tilde{W}_{k',k} \tag{37}$$
for all triples k'', k', k.

Note here that $\tilde{W}_{k',k}$ is defined on the pair $\mathcal{FB}^X_{k,(m,h),g}$ and not just on $\mathcal{F}^X_{k,(m,h)}$. This expresses the requirement that $\tilde{W}_{k',k}$, as an operator on the Fock space $\mathcal{F}^X_{k,(m,h)}$, takes basis states in $\mathcal{B}^X_{k,(m,h),g}$ to basis states in $\mathcal{B}^X_{k',(m,h),g}$.

For $X = N, I$, $\tilde{W}_{k',k}$ can be replaced by a simpler index independent operator $\tilde{W}$ on $\mathcal{FB}^X_{k,g} = (\mathcal{F}^X_k, \mathcal{B}_{k,g})$ which increases the base by 1 unit. For each base k,
$$|\gamma, m, s', l', u'\rangle_{k+1,g} = \tilde{W}|\gamma, m, s, l, u\rangle_{k,g} \tag{38}$$
is a state of a string of base $k+1$ qukits. It represents the same number as does $|\gamma, m, s, l, u\rangle_k$, which is a state of a string of base k qukits. Note that $\tilde{W}$ is not unitary. It is an isometry or unilateral shift [25] because of the lower limit of $k \geq 2$. One sees that iteration of the action of $\tilde{W}$ on $\mathcal{FB}^X_{2,g}$ generates all spaces $\mathcal{FB}^X_{k,g}$. However, $\tilde{W}^\dagger\mathcal{FB}^X_{2,g} = 0$. It follows that $\tilde{W}$ is the generator of a semigroup of transformations on $\mathcal{FB}^X_g$.

This simple description does not extend to $\mathcal{FB}^{Ra}_{k,g}$ as $\tilde{W}$ is not defined on $\mathcal{FB}^{Ra-I}_{k,g}$, the noninteger part of $\mathcal{FB}^{Ra}_{k,g}$. Instead one has to restrict consideration to the indexed base change operators $\tilde{W}_{k',k}$.

The domain and range of $\tilde{W}_{k',k}$ depend on the relation between the prime factors of k and k'. If k and k' have no common prime factors, then $\tilde{W}_{k',k}$ is not defined on the noninteger part $\mathcal{FB}^{Ra-I}_{k,g}$ of $\mathcal{FB}^{Ra}_{k,g}$. It is defined on the integer subspaces of $\mathcal{F}B^{Ra}_{k,g}$ and $\mathcal{F}B^{Ra}_{k',g}$ and is an arithmetic isomorphism (a unitary operator that preserves arithmetic relations and operations) on the subspaces. It satisfies
$$\tilde{W}_{k',k}\mathcal{F}B^{Ra}_{k,g}|_{int} = \mathcal{FB}^{Ra}_{k',g}|_{int}. \tag{39}$$

For cases in which k and $k\prime$ have prime factors in common, the domain and range of $\tilde{W}_{k',k}$ includes some noninteger states in $\mathcal{FB}^{Ra}_{k,g}$. The different cases can be summarized as follows: Let $PF(k)$ denote the prime factors of k. Then

$$\begin{array}{l}
\text{If } PF(k) \subset PF(k') \text{ then } \tilde{W}_{k',k}\mathcal{FB}^{Ra}_{k,g} \subset \mathcal{FB}^{Ra}_{k',g}, \\
\text{If } PF(k) \supset PF(k') \text{ then } \tilde{W}_{k',k} \subset \mathcal{FB}^{Ra}_{k,g} = \mathcal{FB}^{Ra}_{k',g}, \\
\text{If } PF(k), PF(k') \text{ each have elements not in the other and} \\
\qquad \text{share elements in common, then } \tilde{W}_{k',k} \subset \mathcal{FB}^{Ra}_{k,g} =\subset \mathcal{FB}^{Ra}_{k',g}, \\
\text{If } PF(k) = PF(k') \text{ then } \tilde{W}_{k',k}\mathcal{FB}^{Ra}_{k,g} = \mathcal{FB}^{Ra}_{k',g}.
\end{array} \tag{40}$$

In the above $\subset \mathcal{FB}^{Ra}_{k,g}$ denotes a subspace of $\mathcal{FB}^{Ra}_{k,g}$. In all these cases, if the state $|\gamma,(m,h),s,l,u\rangle_{k,g}$ is in the domain of $\tilde{W}_{k',k}$, then the base k' state, $\tilde{W}_{k',k}|\gamma,(m,h),s,l,u\rangle_{k,g}$, represents the same rational number as does $|\gamma,(m,h),s,l,u\rangle_{k,g}$.

The case where $PF(k) = PF(k')$ is of special interest because for each k there is a smallest k' that has the same prime factors as k. If

$$k = p_{j_1}^{h_1} \cdots p_{j_n}^{h_n}, \tag{41}$$

then the smallest k' is given by

$$k' = p_{j_1} \cdots p_{j_n}. \tag{42}$$

Here p_{j_a} for $a = 1, 2, \cdots, n$ is the $j_a th$ prime number.

A special example of this consists of the sets of k that have the first n primes as their factors for $n = 1, 2, \cdots$. Define k_n by

$$k_n = p_1 p_2 \cdots p_n = 2 \times 3 \times \cdots \times p_n. \tag{43}$$

Then the basis states $|\gamma, m, s, l, u\rangle k_n, g$ in $\mathcal{F}B_{k_n,g}$ represent the same numbers as do the basis states in $\mathcal{FB}_{k',g}$ where k' is any base that has the same prime factors as k_n.

As might be expected the group multiplication properties of $\tilde{W}_{k',k}$ depend on the relation between the prime factors of k and k'. Let $[k]$ be the set of all k' that have the same prime factors as k. Then, for all k, k', k'' in $[k]$,

$$\tilde{W}_{k'',k} = \tilde{W}_{k'',k'} \tilde{W}_{k',k}. \tag{44}$$

and

$$\tilde{W}^{\dagger}_{k',k} = \tilde{W}_{k,k'}. \tag{45}$$

Note also that $\tilde{W}_{k,k}$ is the identity on $\mathcal{F}^{Ra}_{k}$.

These properties of the $\tilde{W}_{k',k}$ for different k, k' mimic the corresponding properties of subsets of rational numbers expressed as finite digit strings in any base k. To see this one notes that the set, Ra_k, of rational numbers expressible by states of finite strings of base k qukits is also representable by the set of numbers i/k^n for $n = 0, 1, \cdots$ where i is any integer such that if $n > 0$, then i does not have k as a factor. This follows from the observation that, for any rational number, one can always shift the $k - al$ point to the right hand end by multiplying by a power of k. For example, $97.31 = 9731. \times 0.01$.

One sees that the relations given for the $\tilde{W}_{k',k}$ also apply to the different Ra_k. If k and k' have no common prime factors, then the integers are the only rational numbers that Ra_k and $Ra_{k'}$ have in common. If k has prime factors not in k' and all prime factors of k' are factors of k, then $Ra_{k'} \subset Ra_k$. If k and k' have the same prime factors then $Ra_k = Ra_{k'}$.

3.3 Gauge Transformations

So far all components of the transformation $\mathcal{FB}^X_{k,(m,h),g} \to \mathcal{FB}^X_{k',(m',h'),g'}$ have been treated except the change of gauge or basis from g to g'. This is done by means of gauge transformations U_k. Here U_k is defined as a $U_1 \times SU(k)$ valued function

$$U_k : I \times I \to U_1 \times SU(k) \tag{46}$$

on $I \times I$. U_k is

$$\begin{array}{c} \text{global if } U_k(i,j) \text{ is independent of } (i,j) \\ \text{local if } U_k(i,j) \text{ depends on } (i,j). \end{array} \tag{47}$$

Gauge transformations are different from the base change and shift transformations in that they have no counterpart in classical representations of rational numbers as finite strings of digits in a base k. Unlike the other transformations, which apply to both classical kit strings and quantum qukit strings, gauge freedom and the associated transformations from one gauge to another are strictly quantum theoretical. This follows from the observation that the choice of a gauge corresponds to the choice of a basis in the k dimensional Hilbert space of states for each qukit integer location.

In many situations the choice of gauge is fixed. It plays no role in the representation of states and dynamics of qukits. However there are other situations, such as those occurring in quantum cryptography [22, 26, 27] where rotations of the axis, or gauge transformations, play an important role. Gauge transformations also play a role in the construction of decoherence free subspaces for reference frame changes in quantum information [28, 29, 30] and in gauge theories [31, 32].

The effect of gauge transformations U_k on a state $|\gamma, m, s, l, u\rangle_{k,g}$ of a string of base k qukits is given by

$$\begin{aligned} U_k|\gamma, m, s, l, u\rangle_{k,g} &= c^\dagger_{\gamma,m} U_k(u)(a^\dagger_k)_{s(u),u} \cdots U_k(l)(a^\dagger_k)_{s(l),l}|0\rangle \\ &= (c^\dagger_{\gamma,m}((a^\dagger_k)_{U_k(u)})_{s(u),u} \cdots ((a^\dagger_k)_{U_k(l)})_{s(l),l}|0\rangle \end{aligned} \tag{48}$$

where

$$\begin{aligned} ((a^\dagger_k)_{U_k(j)})_{\alpha,j} &= U_k(j)(a^\dagger_k)_{\alpha,j} = \textstyle\sum_\beta U_k(j)_{\alpha,\beta}(a^\dagger_k)_{\beta,j} \\ ((a_k)_{U_k(j)})_{\beta,j} &= (a_k)_{\beta,j}U^\dagger_k(j) = \textstyle\sum_\alpha U^*_k(j)_{\alpha,h}a_{\alpha,j} \end{aligned} \tag{49}$$

These results are based on the representation of $U_k(j)$ as

$$U_k(j) = \sum_{\alpha,\beta}(U_k(j))_{\alpha,\beta}(a^\dagger_k)_{\alpha,j}(a_k)_{\beta,j}. \tag{50}$$

The parameter h is suppressed in the above as it is the same for each qukit A-C operator. Thus $U_k(j)$ denotes $U_k(j,h)$ which is an element of $U(k) = U(1) \times SU(k)$ and $((a^\dagger_k)_{U_k(j)})_{\alpha,j}$ denotes $((a^\dagger_k)_{U_k(j,h)})_{\alpha,(j,h)}$. Also from now on the m subscript will be often suppressed unless it it needed to help in understanding. Note that, in the interest of simplicity, gauge transformations for the sign qubit

are not considered here. Adding them by including $U_2(m)$ as an element of $U_1 \times SU(2)$ adds nothing new.

The need for the subscript g is evident now in that one has for any state $|\gamma, m, s, l, u\rangle_{k,g}$

$$U_k|\gamma, m, s, l, u\rangle_{k,g} = |\gamma, m, s, l, u\rangle_{k,g'}. \tag{51}$$

Without the subscript change $g \to g'$ there would be no way to show that the two states are different. Note that the state $|\gamma, m, s, l, u\rangle_{k,g'}$ is a linear superposition of the states $|\gamma, m, s, l, u\rangle_{k,g}$

$$|\gamma, m, s, l, u\rangle_{k,g'} = \sum_{\gamma', s'} |\gamma', m, s', l', u'\rangle_{k,g} \langle\gamma', m, s', l', u'|U_k|\gamma, m, s, l, u\rangle_{k,g}. \tag{52}$$

Here the s' sum includes a sum over l', u'.

A single qubit example of these equations is given by $|i\rangle_{g'} = U_2|i\rangle_g$ for $i = 0, 1$ where $U_2 = 1/\sqrt{2} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}$. Then $|1\rangle_{g'} = |+\rangle_g = 1/\sqrt{2}(|1\rangle_g + |0\rangle_g)$ and $|0\rangle_{g'} = |-\rangle_g = 1/\sqrt{2}(|1\rangle_g - |0\rangle_g)$.

Arithmetic relations and operators transform in the expected way. For the relations one defines $=_{A,k,g'}$ and $\leq_{A,k,g'}$ by

$$\begin{array}{c} =_{A,k,g'} := (U_k =_{A,k,g} U_k^\dagger) \\ \leq_{A,k,g'} := U_k \leq_{A,k,g} U_k^\dagger. \end{array} \tag{53}$$

These relations express the fact that $U_k|\gamma, s\rangle_{k,g} =_{A,k,g'} U_k|\gamma', s'\rangle_{k,g}$ if and only if $|\gamma, s\rangle_{k,g} =_{A,k,g} |\gamma', s'\rangle_{k,g}$. Also $U_k|\gamma, s\rangle_{k,g} \leq_{A,k,g'} U_k|\gamma', s'\rangle_{k,g}$ if and only if $|\gamma, s\rangle_{k,g} \leq_{A,k,g} |\gamma', s'\rangle_{k,g}$. Here and below the subscripts g and g' have been added to denote which basis is used to define the arithmetic relations, operations, and number basis sets.

For any of the operations $\tilde{O}_{A,k,g}$ where $O = +, -, \times, \div_\ell$, one defines $\tilde{O}_{A,k,g'}$ by

$$\tilde{O}_{A,k,g'} := (U_k \times U_k \times U_k)\tilde{O}_{A,k,g}(U_k^\dagger \times U_k^\dagger). \tag{54}$$

Then

$$\begin{array}{l} \tilde{O}_{A,k,g'}|U_k(\gamma, s)\rangle_{k,g'}|U_k(\gamma', s')\rangle_{k,g'}) \\ \quad = |U_k(\gamma, s)\rangle_{k,g'}|U_k(\gamma', s')\rangle_{k,g'}|(U_k(\gamma, s))O_{A,k,g'}(U_k(\gamma' s'))\rangle_{k,g'} \end{array} \tag{55}$$

where

$$\begin{array}{l} |U_k(\gamma, s)\rangle_{k,g'} = U_k|\gamma, s\rangle_{k,g} \\ |U_k(\gamma', s')\rangle_{k,g'} = U_k|\gamma', s'\rangle_{k,g} \\ |(U_k(\gamma, s))O_{A,k,g'}(U_k(\gamma' s'))\rangle_{k,g'} = U_k|(\gamma, s)O_{A,k,g}(\gamma'.s')\rangle_{k,g}. \end{array} \tag{56}$$

These equations, which are based on Eq. 18, show the transformations of the basic arithmetic operations on g gauge states to those on g' gauge states.

The different number of U_k and $U_k^\dagger$ factors arises because the operations $\mathcal{O}_{A,k,g}$, acting on two q_k string states, create a third q_k string state. For this paper it is immaterial whether this refers to creation of new $q_k' s$ or to transfer from some supply of $q_k' s$. In the latter case the total number of q_k is preserved.

3.4 Commutation Relations

It is of interest to investigate the commutation relations between $\tilde{T}_1,\ \tilde{T}_2,\ \tilde{W}_{k',k}$, and U_k. The simplest is between $\tilde{T}$ and $\tilde{W}_{k,k'}$ in that these commute:

$$\tilde{T}_i\tilde{W}_{k',k} - \tilde{W}_{k',k}\tilde{T}_i = 0 \tag{57}$$

for $i = 1,2$. Because $\tilde{W}_{k',k}$ is not defined for all k, k', this makes sense only in cases where this operator is defined. Also $\tilde{T}_1$ commutes with $\tilde{T}_2$

The commutation relation for $\tilde{T}_i$ and U_k is straightforward. One has

$$\tilde{T}_i U_k - U^i_k \tilde{T}_i = 0. \tag{58}$$

Here U^i_k for $i = 1,2$ is defined by $U^1_k(j+1,h) = U_k(j,h)$ and $U^2_k(j,h+1) = U_k(j,h)$. This shows that $\tilde{T}_i$ and U_k commute if and only if U_k is global in the *ith* index.

The problems come when one attempts to give a commutation relation between $\tilde{W}_{k',k}$ and U_k. It appears that this is possible if and only if k' is a power of k, such as $k' = k^n$. The reason this works is that there is a one-one map from the set of all $n - tuples$ of base k digits onto the set of base k' digits. As a simple example let $k = 2$ and $k' = 8$. Then there is a one-one map between the triples $000, 001, \cdots, 111$ and $0, 1, \cdots, 7$. For $k' = k^n$ the commutation relation is

$$\tilde{W}_{k',k}U_k - U'_{k'}\tilde{W}_{k',k} = 0. \tag{59}$$

Here each element of $U'_{k'}$ is a product of successive $n - tuples$ of elements of U_k. One has (h is suppressed as it is the same everywhere)

$$\begin{aligned} U'_{k'}(m-j) = \quad & U_k(m - n(j-1) - 1) \times U_k(m - n(j-1) - 2)\times \\ & \quad \cdots \times U_k(m - nj) \text{ for } j \geq 1 \\ U'_{k'}(m+j) = \quad & U_k(m + n(j+1) - 1) \times U_k(m - n(j+1) - 2)\times \\ & \quad \cdots \times U_k(m + jn) \text{ for } j \geq 0. \end{aligned} \tag{60}$$

Even though there do not seem to be commutation relations between U_k and $\tilde{W}_{k',k}$ for arbitrary k and k', one can always use the gauge transformations to define transformed base change operators. For any pair, $U_k,\ U_{k'}$, of gauge transformations, the map $(\tilde{W}_{U',U})_{k',k}$, defined by

$$(\tilde{W}_{U',U})_{k',k} = U_{k'}\tilde{W}_{k',k}U^\dagger_k, \tag{61}$$

is a number preserving map between gauge transformed states just as $\tilde{W}_{k',k}$ is between the original states. One has

$$\begin{aligned} &(\tilde{W}_{U',U})_{k',k}U_k|\gamma, m, s, l, u\rangle_{k,g} \\ &\quad = U_{k'}|\gamma, m, s', l', u'\rangle_{k',g} = U_{k'}\tilde{W}_{k',k}|\gamma, m, s, l, u\rangle_{k,g}. \end{aligned} \tag{62}$$

Here $U_{k'}|\gamma, m, s', l', u'\rangle_{k',g}$ represents the same number in the gauge transformed k' basis as $U_k|\gamma, m, s, l, u\rangle_{k,g}$ does in the gauge transformed k basis. Note that

the restrictions on the domain and range for $\tilde{W}_{k',k}$ in the original representation apply to the domain and range of $(\tilde{W}_{U',U})_{k',k}$ in the transformed representation.

The meaning of the statements "...the same number as..." is based on the transformed basic arithmetic relations and operations, given by Eqs. 53-55, in both the k and k' bases. The number represented by the state $U_k|\gamma, m, s, l, u\rangle_{k,g}$ is determined by its properties relative to the transformed relations and operations $\tilde{=}_z, \tilde{\leq}_z, \tilde{+}_z, \tilde{-}_z, \tilde{\times}_z, \tilde{\div}_{z,\ell}$ where the subscript z denotes the 4-tuple A, k, m, g'. Similarly the number represented by $U_{k'}|\gamma, m, s', l', u'\rangle_{k'}$ is determined by properties based on the basic relations where z denotes the 4-tuple A, k', m, g'. These two numbers should be the same.

This emphasizes that one must also transform the relations and operations along with the state. The reason is that as Eq. 52 shows, relative to the untransformed relations and operations, the states $U_k|\gamma, m, s, l, u\rangle_{k,g}$ and $U_{k'}|\gamma, m, s', l', u'\rangle_{k',g}$ are linear superpositions of states representing many different numbers.

3.5 $k = 1$

So far all number bases have been considered except one, the value $k = 1$. The $k = 1$ string representations are called unary representations. These are not usually considered, because basic arithmetic operations on these numbers are exponentially hard. For instance the number of steps needed to add two unary numbers is proportional to the values of the numbers and not to the logarithms of the values. However, even though they are not used arithmetically, they are always present in an interesting way.

To see this one notes that $k = 1$ representations are the only ones that are extensive, all others are representational. The representational property for $k \geq 2$ base states of a qukit string means that a number represented by a state has nothing to do with the properties of the string state. The number represented by the state, $|672\rangle$, of a string of 3 $q'_{10}s$ is unrelated to the properties of the qukits in the state.

Extensivity of any unary representation means that any collection of systems is an unary representation of a number that is the number of systems in the collection. There are many examples. A system of spins on a lattice is an unary representation of a number, that is the number of spins in the system. A gas of particles in a box is an unary representation of a number, that is the number of particles in the box. The qukit strings that play such an important role in this paper are unary representations of numbers, that are the number of qukits in the strings. A single qukit is an unary representation of the number 1.

This omnipresence of unary representations relates to another observation that 1 is the only number that is a common factor of all prime numbers and of all numbers. So it is present as a factor of any base. This ties in with the observation that unary representations of rational numbers as a single collection of systems do not exist.[6] Natural numbers and integers are the only ones with

[6] Pairs of unary representations will work since all integers can be so represented. But

unary representations. This ties in nicely with the observation that, for any pair k, k', the domain of $\tilde{W}_{k',k}$ includes the integer subspace of states, $\mathcal{FB}^I_{k,m,g}$ in $\mathcal{FB}^{Ra}_{k,m,g}$, and if k, k' have no prime factors in common, $\mathcal{FB}^I_{k,m,g}$ and $\mathcal{FB}^I_{k',m,g}$ are the domain and range of $\tilde{W}_{k',k}$.

The extensivity of unary representations supports the inclusion of the $U(1)$ factor, Eq. 46, in the definition of gauge transformations. A state of $(a^\dagger_k)_{\alpha,(i,j)}|0\rangle$ of a qukit in state $|\alpha\rangle$ at location (i, j) is also an unary representation of the number 1. Multiplication of this state by a phase factor $e^{i\theta_{i,j}}$ is a transformation that gives another state that is also an unary representation of the number of qukits represented by the state.

This argument extends to states of strings of qukits. A phase factor associated with any state of a string of q_k at sites $(l, h), \cdots (u, h)$ is a product of the phase factors associated with each of the q_k in the string. If $e^{i\theta_{j,h}}$ is a phase factor for q_k at site (j, h), then $e^{i\Theta_{[(l,h),(u,h)]}}$, where $\Theta_{[(l,h),(u,h)]} = \sum_{j=l}^{u} \theta_{j,h}$, is the phase factor for any state of the string.

As is well known, multiplying any state by a phase factor gives the same state as far as any physical meaning is concerned. However here one can have linear superpositions of states of strings of q_k both at different locations and of different length strings. In these cases the phase factors do matter as they change the relative phase between the components in the superposition.

4 Symmetries and Invariances

So far it has been seen that the spaces of quantum representations of N, I, and Ra are parameterized by the location, (m, h), of the qukit strings, the base k, and the choice of gauge or basis in the k dimensional Hilbert space of states for each q_k location in $I \times I$. Associated with this parameterization are the operators on the corresponding representation space: translation operators $\tilde{T}_1, \tilde{T}_2$, base change operators $\tilde{W}_{k',k}$, and (base dependent) gauge transformations U_k.

Of interest are the symmetry or invariance aspects of various properties and operations for the qukit strings and their states. In particular, one would like to know which properties are invariant under all the transformation operators and which are not. Invariant properties can also be regarded as symmetries of the space.

One set of invariant properties are those expressed by the axioms and theorems for each number type. This is a direct consequence of the fact that each axiom and theorem expresses a property which is valid for all representations in the space. In other words, the truth value of each axiom and theorem is unchanged under any of the transformations on the representation space. Each axiom and theorem is true for all representations in the space.

Another way to express this in more physical terms is to call a property invariant if it is conserved as one "moves" the qukit string around in the representation space. This includes changes in the locations of the string on $I \times I$,

rational numbers as integer pairs are not being considered here.

changes in the base, and changes in the basis for the states of each qukit. However there are differences. In particular, the change of base corresponds to a change from strings of one q_k system to strings of another $q_{k'}$ system. Base k qukits are different from base k' qukits just as a spin S system is different from a spin S' system.

It should be emphasized again that the invariance of axioms and theorems under transformations in the representation space is not trivial and obvious. Each representation in the space describes qukit strings and their states for a specific $I \times I$ location, (m, h), a specific base, k, and a gauge choice, g for the states of all qukit locations. By themselves, the states of these systems cannot be said to represent numbers of any type. The validity of the statement that these states represent numbers is based on

- defining operations and relations on the states of the qukit strings in terms of basic operations on the states of the individual qukits,
- proving that these relations and operations satisfy the axioms and theorems for the number type being considered.

In particular, arithmetic relations, $=_{A,k,(m,h),g}$, $\leq_{A,k,(m,h),g}$, and operations, $\tilde{+}_{A,k,(m,h),g}$, $\tilde{\times}_{A,k,(m,h),g}$, for the N, I, and Ra spaces, $\tilde{-}_{A,k,(m,h),g}$ for the I and Ra spaces and $\tilde{\div}_{A,k,(m,h),g,\ell}$ are defined in terms of basic properties and operations on the q_k in the strings and their states. Here this would consist of a definition of these relations and operations in terms of sums of products of qukit A-C operators. Then one would have to prove that these relations and operations satisfy the axioms and theorems for the number type being considered.

In this paper these steps have not been provided. Instead the treatment is more like that in mathematical analysis textbooks in that descriptions of the basic arithmetic relations and operations are more directly based on the required axiomatic properties and not on algorithms for operations on strings of qukits. In essence the treatment here is more like a translation of the properties as described in textbooks into the language of quantum mechanics for strings of qukits and assuming that the relevant proofs apply in this case also. A full treatment would require first detailed definitions of the basic arithmetic operations and then proofs that they satisfy the appropriate axioms. Some details are given in [20, 21].

The invariance of axioms and theorems under transformations should be distinguished from the covariance of their expressions in various representations. Consider for example the axiom $x + 0 = x$ which says that 0 is the additive identity. The expression of this axiom in the space $\mathcal{FB}^{X}_{k,m,g}$ for $X = N, I, Ra$ is (again h is suppressed)

$$\begin{aligned} &\tilde{+}_{A,k,m,g}|\gamma, m, s, l, u\rangle_{k,g}|+, m, 0\rangle_{k,g} \\ &\quad = |\gamma, m, s, l, u\rangle_{k,g}|+, m, 0\rangle_{k,g}|\gamma, m, s, l, u\rangle_{k,g} \end{aligned} \tag{63}$$

for all γ, s. Here $|+, m, 0\rangle_{k,g} = c^{\dagger}_{+,m}(a^{\dagger}_{k})_{0,m}|0\rangle$ is a base k qukit state for the number zero.

Under the action of $\tilde{T}_1$ this axiom expression becomes

$$\begin{aligned}\tilde{+}_{A,k,m+1,g}&|\gamma, m+1, s', l+1, u+1\rangle_{k,g}|+, m+1, 0\rangle_{k,g} \\ &= |\gamma, m+1, s', l+1, u+1\rangle_{k,g}|+, m+1, 0\rangle_{k,g} \\ &\quad \times|\gamma, m+1, s', l+1, u+1\rangle_{k,g}.\end{aligned} \tag{64}$$

Here $s'(j+1) = s(j)$.

For $\tilde{W}_{k',k}$ the axiom expression is

$$\begin{aligned}\tilde{+}_{A,k',m,g}&\tilde{W}_{k',k}|\gamma, m, s, l, u\rangle_{k,g}\tilde{W}_{k',k}|+, m, 0\rangle_{k,g} \\ &= \tilde{+}_{A,k',m,g}|\gamma, m, s'', l', u'\rangle_{k',g}|+, m, 0\rangle_{k',g} \\ &= |\gamma, m, s'', l', u'\rangle_{k',g}|+, m, 0\rangle_{k',g}|\gamma, m, s'', l', u'\rangle_{k',g}\end{aligned} \tag{65}$$

This makes sense only for k', k for which $\tilde{W}_{k',k}$ is defined. For these the state $|\gamma, m, s'', l', u'\rangle_{k',g}$ represents the same number as does $|\gamma, m, s, l, u\rangle_{k,g}$.

For U_k the expression of the axiom is unchanged except that it refers to a different basis or reference frame. Use of Eq. 51 gives

$$\begin{aligned}\tilde{+}_{A,k,m,g'}&U_k|\gamma, m, s, l, u\rangle_{k,g}U_k|+, m, 0\rangle_{k,g} \\ &= \tilde{+}_{A,k,m,g'}|\gamma, m, s, l, u\rangle_{k,g'}|+, m, 0\rangle_{k,g'} \\ &= |\gamma, m, s, l, u\rangle_{k,g'}|+, m, 0\rangle_{k,g'}|\gamma, m, s, l, u\rangle_{k,g'}.\end{aligned} \tag{66}$$

Recall from Eq. 49 that the AC operators for the states in the changed basis are linear superpositions of the AC operators for the states in the original basis.

The invariance of the axioms and theorems should be distinguished from the quantum mechanical property of conservation of probability for any unitary transformation. For any state ψ, unitary transformation V and property expressed by a projection operator P, one always has $(V\psi, P_V V\psi) = (\psi, P\psi)$ where $P_V = VPV^\dagger$. However the property expressed by P is invariant with respect to V if and only if P commutes with V, $PV = VP$. It follows that if P expresses any of the axioms, then P should commute with $\tilde{T}_1, \tilde{T}_2, \tilde{W}_{k',k}, U_k$.

An example of a property that is not invariant is $=_{A,k,m,g}$. This is used to express arithmetic equality in axioms such as "$x + 0 = x$". The problem is that the corresponding projection operator, $\tilde{P}_{=_{A,k,m,g}}$, defined by Eq. 13 is not invariant under the action of $\tilde{T}_1, \tilde{W}_{k',k}$, or U_k. One has

$$\begin{aligned}\tilde{T}_1\tilde{P}_{=_{A,k,m,g}} &= \tilde{P}_{=_{A,k,m+1,g}}\tilde{T}_1 \\ \tilde{W}_{k',k}\tilde{P}_{=_{A,k,m,g}} &= \tilde{P}_{=_{A,k',m,g}}\tilde{W}_{k',k} \\ U_k\tilde{P}_{=_{A,k,m,g}} &= \tilde{P}_{=_{A,k,m,g'}}U_k.\end{aligned} \tag{67}$$

Part of the invariance lack is due to the fact that $=_{A,k,m,g}$ and the corresponding projection operator $\tilde{P}_{=_{A,k,m,g}}$ are locally defined. Invariance with respect to $\tilde{T}_1$ can be obtained by an expanded definition of arithmetic equality $=_{A,k,g}$ given by $=_{A,k,g}\leftrightarrow \exists m(=_{A,k,m,g})$ with a corresponding projection operator

$$\tilde{P}_{=_{A,k,g}} = \sum_m \tilde{P}_{=_{A,k,m,g}}. \tag{68}$$

(Eq. 13 with the sum over h, h' shows that the operator is invariant under $\tilde{T}_2$.)

One would like to complete the process by expanding the definitions of arithmetic relations and operations to be invariant under $\tilde{W}_{k,k'}$ and U_k. However there are problems in that U_k depends on k. This precludes use of a summation over k because this causes problems with the property that the gauge transformations U_k depend on k.

To see the problem one notes that although $P_{=_{A,k,g}}$ is invariant under $\tilde{T}_1, \tilde{T}_2$, it is not invariant under either $\tilde{W}_{k',k}$ or U_k. The lack of U_k invariance follows from Eq. 14 which shows that

$$\begin{aligned} U_k \tilde{P}_{\gamma,[s],k,h,g} U_k^\dagger &= U_k \sum_{s' \sim_0 s} \tilde{P}_{|\gamma,(m,h),s',l,u\rangle_{k,g}} U_k^\dagger \\ &= \sum_{s' \sim_0 s} \tilde{P}_{U_k|\gamma,(m,h),s',l,u\rangle_{k,g}} = \tilde{P}_{\gamma,[s],k,h,g'}. \end{aligned} \tag{69}$$

Here the g and g' subscripts have been inserted to show the differences in the projection operators.

Two approaches to this problem seem possible. One is based on the use of gauge invariant representations of qukit states based on irreducible representations of $SU(k)$ [28, 29, 30]. The other is based on the possible use of gauge theory [31, 32] to express the invariance of the arithmetic properties by means of an action. Further investigation of this problem will be left to future work.

5 Composite and Elementary Qukits

So far the qukit components of strings are considered to be different systems for each value of k. A k qukit is different from a k' qukit just as a spin k system is different from a spin k' system. This leads to a large number of different qukit types, one for each value of k. However, the dependence of the properties of the base changing operator $\tilde{W}_{k',k}$ on the prime factors of k and k' suggests that one consider qukits q_k as composites q_{c_k} of prime factor qukits q_{p_n}. In general the relation between the base k q_k and the composite base k q_{c_k} is given by

$$q_{c_k} = q_{p_{j_1}}^{h_1} q_{p_{j_2}}^{h_2} \cdots q_{p_{j_n}}^{h_n}. \tag{70}$$

where (Eq. 41)

$$k = p_{j_1}^{h_1} p_{j_2}^{h_2} \cdots p_{j_n}^{h_n}$$

. Simple examples of this for $k = 10$ and 18 are $q_{c_{10}} = q_2 q_5$ and $q_{c_{18}} = q_2 q_3 q_3$.

The observation that for each k there is a smallest k' with the same prime factors and its relevance to the properties of $\tilde{W}_{k',k}$ suggest the importance of the $q_{c_{k'}}$ where the powers of the prime factors are all equal to 1(Eq. 42)

$$q_{c_{k'}} = q_{p_{j_1}} q_{p_{j_2}} \cdots q_{p_{j_n}}. \tag{71}$$

A particular example of this for k_n, the product of the first n prime numbers, is shown by (Eq. 43)

$$q_{c_{k_n}} = q_2 q_3 q_5 \cdots q_{p_n} \tag{72}$$

.

These considerations suggest a change of emphasis in that one should regard prime number qukits q_{p_n} as basic or elementary and the qukits q_k as composites of the elementary ones. In this case one would want to consider possible physical properties of the elementary qukits and how they interact and couple together to form composites. This is a subject for future work as the emphasis here is on arithmetic properties, not on physical properties. It is, however, intriguing to note that if the prime number q_{p_n} are considered as spin systems with spin s_n given by $2s_n + 1 = p_n$, then there is just one fermion, q_2. All the others are bosons.

As was the case for strings of q_k, one wants to represent numbers by states of finite strings of composite q_{c_k}. In general, this involves replacing the k dimensional Hilbert space $\mathcal{H}_k$ at each site in $I \times I$ by a product space

$$\mathcal{H}_{c_k} = \mathcal{H}^{h_1}_{p_{j_1}} \otimes \cdots \otimes \mathcal{H}^{h_n}_{p_{j_n}} \tag{73}$$

and then following the development in the previous sections to describe number states. In particular the gauge fixing would apply to each component space in Eq. 73 for each location in $I \times I$.

In the following, consideration will be limited to the simpler case where all the powers $h_i = 1$ as in Eq. 71. In addition the elementary q_{p_j} in $q_{c_{k'}}$ at each site in $I \times I$ will be considered noninteracting. In this case the AC operators $(a^{\dagger}_{k'})_{\alpha,(j,h)}$ $(a_{k'})_{\alpha,(j,h)}$ with $\alpha = 0, 1, \cdots, k'-1$ for $q_{k'}$ at site (j,h) in state α are replaced by products of AC operators for the component elementary q_{p_j}. One has

$$\begin{aligned} (a^{\dagger}_{k'})_{\alpha,(j,h)} &= V_{k'}(a^{\dagger}_{p_{j_1}})_{d_1,(j,h)}(a^{\dagger}_{p_{j_2}})_{d_2,(j,h)} \cdots (a^{\dagger}_{p_{j_n}})_{d_n,(j,h)} \\ (a_{k'})_{\alpha,(j,h)} &= (a_{p_{j_1}})_{d_1,(j,h)}(a_{p_{j_2}})_{d_2,(j,h)} \cdots (a_{p_{j_n}})_{d_n,(j,h)} V^{\dagger}_{k'} \end{aligned} \tag{74}$$

where for each $i = 1, \cdots, n$ d_i is an element of $0, 1, \cdots, p_{j_i} - 1$ and

$$\alpha = \beta(d_1, \cdots, d_n). \tag{75}$$

Here $\beta : \{\{0, \cdots, p_{j_1} - 1\} \times \cdots \times \{0, \cdots, p_{j_n} - 1\}\} \to \{0, \cdots, k' - 1\}$ maps states of the $n - tuple$ of prime number qukits of $q_{c_{k'}}$ to states of $q_{k'}$. $V_{k'}$ is a unitary operator that maps states in $\mathcal{H}_{p_{j_1}} \times \cdots \mathcal{H}_{p_n}$ to states in $\mathcal{H}_{k'}$. For any q_{c_k} in state $|d_1, \cdots, d_n\rangle$, $V_k|d_1, \cdots, d_n\rangle = |\alpha\rangle$.

The replacements given above can be used for states of strings of composite $q_{c_{k'}}$ systems. Each state $|\gamma, (m,h), s', l, u\rangle_{c_{k'}}$ is given by strings of AC operators as

$$\begin{aligned} &|\gamma, (m,h), s', l, u\rangle_{c_{k'}} = c^{\dagger}_{\gamma,(m,h)}\{(a^{\dagger}_{p_{j_1}})_{d'_1(u,h),(u,h)} \cdots (a^{\dagger}_{p_{j_n}})_{d'_n(u,h),(u,h)}\} \\ &\cdots \{(a^{\dagger}_{p_{j_1}})_{d'_1(l,h),(l,h)} \cdots (a^{\dagger}_{p_{j_n}})_{d'_n(l,h),(l,h)}\}|0\rangle. \end{aligned} \tag{76}$$

Here s' is a function from $I \times I$ to $\{\{0, \cdots, p_{j_1}\} \times \cdots \times \{0, \cdots, p_{j_n} - 1\}\}$ and $d_i(j,h)$ is the ith component of $s'(j,h)$.

The requirement that states of the form $|\gamma, (m,h), s', l, u\rangle_{c_{k'},g}$ represent numbers is based on an ordering of the basis states of q_{c_k}, or, what is equivalent, an

ordering of the $n - tuples$ in the range set of s'. The definitions of arithmetic relations and operations for these states must respect the ordering and they must satisfy the relevant axioms and theorems for the type of number being considered.

States of strings of composite qukits and their arithmetic properties can be directly related to states of strings of $q_{k'}$ where $k' = p_{j_1} \cdots p_{j_n}$, Eq. 42. The arithmetic properties of the states $|\gamma, (m,h), s, l, u\rangle_{k',g}$, where s is a function from $I \times I$ with values in $\{0, 1, \cdots, k'-1\}$, and the ordering of the states is based on the map β, Eq. 75 which is required to be order preserving.

Also Eqs. 74 and 76 and the unitarity of $V_{k'}$ should give the result that $|\gamma, (m,h), s, l, u\rangle_{k'}$ and $|\gamma, (m,h), s', l, u\rangle_{c_{k'}}$ represent the same number even though they are very different quantum mechanically. This is a nontrivial requirement. It depends on $V_{k'}$, mappings of the ordering, arithmetic relations and operations on states of $q_{c_{k'}}$ strings to those for states of $q_{k'}$ strings, and proof of the invariance of the relevant axiomatic and theorem properties under the action of $V_{k'}$.

The description of the transformation operations $\tilde{T}_1, \tilde{T}_2, \tilde{W}_{k',k}, U_{k'}$ can be extended to apply to the composite qukit strings. $\tilde{T}_1$ and $\tilde{T}_2$ shift all elementary q_{p_j} in $q_{c_{k'}}$ by one unit in either direction. The base changing operator $\tilde{W}_{c_{k'},c_k}$ changes states of q_{c_k} strings to states of $q_{c_{k'}}$ strings that should represent the same number. Note that the expression of $\tilde{W}_{c_{k'},c_k}$ in terms of sums of products of AC operators will include the annihilation of many component elementary qukits in q_{c_k} and creation of many that are components of $q_{c_{k'}}$.

The description of gauge transformations U_{c_k} applied to states of q_{c_k} is interesting. If q_{c_k} is composed of elementary q_{p_j} as given by Eq. 70, then U_{c_k} is a map from $I \times I$ to elements of $U(p_{j_1})^{h_1} \times \cdots \times U(p_{j_n})^{h_n}$. Here $U(p_{j_i})$ is the unitary group of prime dimension p_{j_i}. For the special cases of Eq. 71 and 72 $U_{c_{k'}}$ and $U_{c_{k_n}}$ take values in $U(p_{j_1}) \times \cdots \times U(p_{j_n})$ and in $U(p_1) \times \cdots \times U(p_n)$ respectively. Since $U(p_j) = U(1) \times SU(p_j)$ the values of $U_{c_{k_n}}$ can be represented as elements of

$$\begin{aligned} &U(1) \times SU(p_1) \times SU(p_2) \times \cdots \times SU(p_n) \\ &\qquad = U(1) \times SU(2) \times SU(3) \times SU(5) \times \cdots \times SU(p_n). \end{aligned} \tag{77}$$

Here the phase factor elements in $U(1)$ for each elementary qukit have been combined into one phase factor for the composite $q_{c_{k_n}}$.

The discussion so far suggests that, as far as quantum theory representations of natural numbers, integers, and rational numbers are concerned, it is sufficient to limit components of gauge transformations to products of elements of $U(1)$ and products of elements of $SU(p)$ groups where p is a prime number. Furthermore it is sufficient that, for each prime p, elements of $SU(p)$ occur at most once in the product. It is also sufficient to limit components to products of the form of Eq. 77 for $n = 1, 2, \cdots$ as these will include representations for all rational numbers.

6 Discussion

There are several additional aspects of the work presented here that would benefit from further discussion. The approach in which one regards q_k qukits for arbitrary k as composites q_{c_k} of elementary prime number qukits q_p where p is a prime number has an interesting property. To see this one recalls that the domain and range of the base changing operator $\tilde{W}_{k',k}$, as a map from $\mathcal{FB}^{Ra}_{k,g}$ to $\mathcal{FB}^{Ra}_{k',g}$, depend on the prime factors of k and k'. It follows that one must be able to determine the prime factors of k and k' to obtain the properties of this operator.

However, as is well known, there is no known classical algorithm for efficiently obtaining the prime factors of an arbitrary large number. The only known efficient algorithm for factorization [33] is quantum mechanical and is based on the exponential speedup possible for quantum computers.

Working with composite q_{c_k} systems avoids the factorization problem completely. In this case one works at the outset with composites containing different numbers of prime components. Determination of the value of k for a given set of prime components is efficient and straightforward. It is not clear if the value of k is even needed with this approach. The possible exception is its use as a label or coding, as in $\tilde{W}_{k',k}$, for arbitrary sets of prime components q_p.

The observation that the only known efficient factoring algorithm is quantum mechanical provides some support for the emphasis in this paper on quantum representations of numbers. The possibility of a deeper connection between quantum computation algorithms and the spaces of quantum representations of numbers presented here is a subject for future work.

It is useful to reemphasize the importance of the requirement that the action of the transformation operators, $\tilde{T}_1, \tilde{T}_2, \tilde{W}_{k',k}, U_k$ on basis states $|\gamma,(m,h),s,l,u\rangle_{k,g}$ preserve the number representing property of the states. For instance,

$$\begin{array}{l} \tilde{W}_{k',k}|\gamma,(m,h),s,l,u\rangle_{k,g} = |\gamma,(m,h),s'.l'.u'\rangle_{k',g} \\ \text{and } U_k|\gamma,(m,h),s,l,u\rangle_{k,g} = |\gamma,(m,h),s,l,u\rangle_{k,g'} \end{array} \tag{78}$$

must represent the same number as does $|\gamma,(m,h),s,l,u\rangle_{k,g}$. Note that the state $|\gamma,(m,h),s,l,u\rangle_{k,g'}$ is different from $|\gamma,(m,h),s,l,u\rangle_{k,g}$ in that it is described using transformed AC operators given in Eqs. 48 and 49.

Verification of this requirement involves showing that the basic arithmetic operations and relations must have the properties described by the relevant axioms and theorems. As noted, this is not obvious because the definitions of the operations and relations are based on algorithms for basic quantum operations on states of q_k or q_{c_k} in strings. Here this problem was bypassed by simply assuming that the arithmetic relations and operations have the requisite properties. Some work in this direction is given in [20, 21].

It is hoped to examine this in more detail in the future. One possible approach is to follow the development of gauge theories [24, 32] and define parallel transport for the property "the same number as". The problem here is that one does not have an action or Lagrangian for the axioms of a theory. Recall that

the axioms and theorems of a theory are invariant under the transformations described here. If such an action could be discovered then its invariance would give the desired result.

It is amusing to note that for strings of composite $q_{c_{k_n}}$ where $q_{c_{k_n}} = q_2 q_3 q_5 \cdots q_{p_n}$, Eq. 72, gauge transformations have the form $U(1) \times SU(2) \times \cdots \times SU(p_n)$, Eq. 77. Invariance of the axioms and theorems under these transformations reminds one of the standard model of physics [34, 35, 36] which requires invariance of a Lagrangian under the gauge transformation given above for $n = 3$. Whether there is any further connection or not remains to be seen.

The emphasis in this paper has been on the quantum representations of numbers in N, I and Ra as mathematical systems. The question arises regarding the connection of these results to physics. Making such a connection is essential if one is to make headway towards a coherent theory of mathematics and physics together. Here an indirect connection is implied by using physical concepts of representations parameterized by points in a parameter set and induced transformations of the representations. The invariance of relevant axioms and theorems under the transformations and the decomposition of q_k into composites q_{c_k} of elementary q_p are also concepts used in physics. Also, limitation of the representations to single strings of qukits is dictated by physical considerations.

These connections are all indirect. A more direct one may be to take advantage of the fact that, as units of quantum information, qukits can represent both physical and mathematical systems. This fact is used extensively in work on quantum computation [22] where states of qubit strings represent both numbers in quantum computers and states of physical systems with associated dynamics that are models of quantum computers ([37] and references cited therein). Future work will tell if this approach is useful.

7 Summary

In this paper spaces of quantum representations of natural numbers, integers, and rational numbers were described. The space is based on a parameterization of the different representations, as states of finite strings of qukits, by $4-tuples$ $k, (m, h), g$ which are elements of a parameter set $N \geq 2 \times I \times I \times \{g\}$. Here (m, h) locates the string in a 2 dimensional integer lattice $I \times I$, $k \geq 2$ is the qukit base, and each g is a gauge fixing function that chooses the qukit basis in the k dimensional Hilbert space of states at each lattice point.

Associated with each point $k, (m, h), g$ is a representation space $\mathcal{FB}^X_{k,(m,h),g}$ consisting of a Fock space $\mathcal{F}^X_{k,(m,h)}$ of states of finite strings of base k qukits at locations (j, h) for all $j \epsilon I$, and a set, $\mathcal{B}_{k,(m,h),g} = B_{2,m,h} \cup \{B_{k,j,h,g} : j \epsilon I\}$, of basis states. For each j $B_{k,j,h,g}$ is a basis for $\mathcal{H}^k_{j,h}$ and $B_{2,m,h}$ is the basis for the sign qubit at (m, h). The states $|\gamma, (m, h), s, l, u\rangle_{k,g}$ in $\mathcal{B}_{k,(m,h),g}$ can be said to span $\mathcal{F}^X_{k,(m,h)}$. These states are described by strings of AC operators acting on the qukit vacuum state $|0\rangle$. For $X = N, I, Ra$ the states represent natural numbers, integers, or rational numbers respectively. Here $\gamma = +, -$ denotes the

sign, (m,h) the location of the "$k-al$" point, and s is a $0,1,\cdots,k-1$ valued function on the lattice domain $[(l,h),(u,h)]$. Also $l \leq m \leq u$.

Transformations on the parameter set induce transformations on the representation space. The properties of the base changing operator, $\tilde{W}_{k',k}$, were seen to depend on the relations between the prime factors of k and k'. Global and local gauge transformations U_k were defined as functions from $I \times I$ to $U(1) \times SU(k)$. These k dependent transformations change the gauge or basis choice for the Hilbert spaces on $I \times I$.

The base k was extended to $k=1$ by considering unary representations of numbers. It was noted that these are unique as they are the only ones that are extensive and not representational. The extensivity is seen by noting that any collection of systems, e.g. strings of q_k, is an unary representation of a number that is the number of systems in the collection. This ties in with the ubiquitous presence of $U(1)$ in gauge transformations for all $k \geq 2$. Also the observation that noninteger rational numbers do not have an unary representation by a *single* string or a *single* collection of systems, was seen to tie in neatly with the properties of $\tilde{W}_{k',k}$.

The invariance of arithmetic properties of the qukit string under the action of the transformation operators was discussed. In particular the properties expressed by the axioms and theorems for each type, N, I, Ra of numbers are invariant. It was noted that, the fact that gauge transformation operators U_k depend on the base, causes problems for implementation of the invariance for the base changing operators $\tilde{W}_{k,k'}$ and gauge transformations. Two possible approaches to the problem, the use of gauge invariant representations of qukit states and the use of gauge theory, were suggested. However, further work on the problem was left to future work.

It was noted that the dependence of the properties of $\tilde{W}_{k',k}$ on the prime number factors of k and k', suggests a change in emphasis from qukits as q_k to composites q_{c_k} of elementary prime number qukits q_p. If $k = (p_{j_1})^{h_1} \cdots (p_{j_n})^{h_n}$, then $q_{c_k} = (q_{p_{j_1}})^{h_1} \cdots (q_{p_{j_n}})^{h_n}$. Here the component elementary q_p in a composites were assumed to be noninteracting. Then states of each q_{c_k} can be represented as $m-tuples$ of states of the component q_p where m is the number of components in q_{c_k}. The special case where $q_{c_{k_n}} = q_2 q_3 q_5 \cdots q_{p_n}$ primes was described, particularly with respect to gauge transformations of the form $U(1) \times SU(2) \times \cdots \times SU(p_n)$.

Finally, several different aspects of the work presented here were discussed. The emphasis was on the many different avenues for future work. In any case it is hoped that the ideas presented here will show that studies of quantum representations of numbers as states of single strings of qukits may be useful and relevant to physics. Numbers *are* the foundations of all physical theories. One may also hope that this work is a useful small step towards a coherent theory of mathematics and physics together.

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